Example 2

The vertical distance covered by a rocket from $t = 8$ to $t = 30$ seconds is given by

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

a) Use the two-segment trapezoidal rule to find the distance covered from $t = 8$ to $t = 30$ seconds.
b) Find the true error, E_t for part (a).
c) Find the absolute relative true error for part (a).

Solution

a) The solution using 2-segment Trapezoidal rule is

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$n = 2$$
$$a = 8$$
$$b = 30$$
$$h = \frac{b-a}{n}$$
$$= \frac{30-8}{2}$$
$$= 11$$

$$I \approx \frac{30-8}{2(2)}\left[f(8) + 2\left\{ \sum_{i=1}^{2-1} f(8+11i) \right\} + f(30) \right]$$
$$= \frac{22}{4}\left[f(8) + 2f(19) + f(30) \right]$$
$$= \frac{22}{4}\left[177.27 + 2(484.75) + 901.67 \right]$$
$$= 11266 \text{ m}$$

b) The exact value of the above integral is

$$x = \int_8^{30} \left(2000 \ln \left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$
$$= 11061 \text{ m}$$

so the true error is

$$E_t = \text{True Value} - \text{Approximate Value}$$
$$= 11061 - 11266$$
$$= -205 \text{ m}$$

c) The absolute relative true error, $|\in_t|$, would then be

$$|\in_t| = \left| \frac{\text{True Error}}{\text{True Value}} \right| \times 100$$
$$= \left| \frac{11061 - 11266}{11061} \right| \times 100$$
$$= 1.8537\%$$

Table 1 Values obtained using multiple-segment trapezoidal rule for

$$x = \int_{8}^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

n	Approximate Value	E_t	$\lvert \in_t \rvert \%$	$\lvert \in_a \rvert \%$
1	11868	-807	7.296	---
2	11266	-205	1.853	5.343
3	11153	-91.4	0.8265	1.019
4	11113	-51.5	0.4655	0.3594
5	11094	-33.0	0.2981	0.1669
6	11084	-22.9	0.2070	0.09082
7	11078	-16.8	0.1521	0.05482
8	11074	-12.9	0.1165	0.03560

Example 3

Use the multiple-segment trapezoidal rule to find the area under the curve

$$f(x) = \frac{300x}{1+e^x}$$

from $x = 0$ to $x = 10$.

Solution

Using two segments, we get

$$h = \frac{10-0}{2} = 5$$

$$f(0) = \frac{300(0)}{1+e^0} = 0$$

$$f(5) = \frac{300(5)}{1+e^5} = 10.039$$

$$f(10) = \frac{300(10)}{1+e^{10}} = 0.136$$

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{ \sum_{i=1}^{n-1} f(a+ih) \right\} + f(b) \right]$$

$$= \frac{10-0}{2(2)}\left[f(0) + 2\left\{ \sum_{i=1}^{2-1} f(0+5) \right\} + f(10) \right]$$

$$= \frac{10}{4}\left[f(0) + 2f(5) + f(10) \right]$$

$$= \frac{10}{4}\left[0 + 2(10.039) + 0.136 \right] \quad = 50.537$$

So what is the true value of this integral?

$$\int_0^{10} \frac{300x}{1+e^x}\,dx = 246.59$$

Making the absolute relative true error

$$|\in_t| = \left|\frac{246.59 - 50.535}{246.59}\right| \times 100$$
$$= 79.506\%$$

Why is the true value so far away from the approximate values? Just take a look at Figure 5. As you can see, the area under the "trapezoids" (yeah, they really look like triangles now) covers a small portion of the area under the curve. As we add more segments, the approximated value quickly approaches the true value.

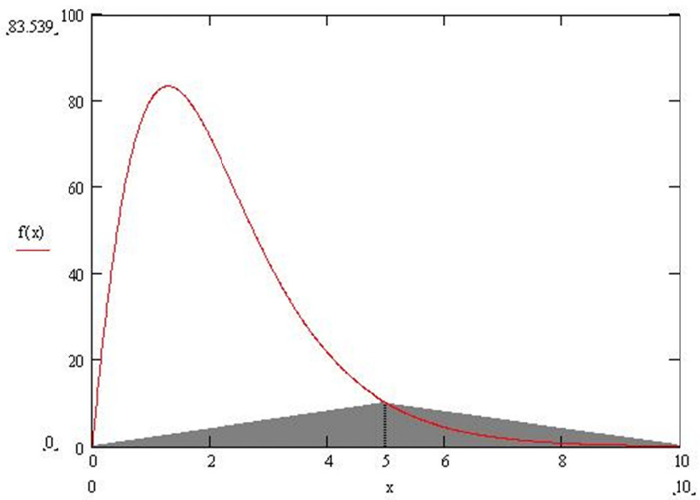

Figure 5 2-segment trapezoidal rule approximation.

Table 2 Values obtained using multiple-segment trapezoidal rule for $\int_0^{10} \frac{300x}{1+e^x}\,dx$.

n	Approximate Value	E_t	$\|\in_t\|$
1	0.681	245.91	99.724%
2	50.535	196.05	79.505%
4	170.61	75.978	30.812%
8	227.04	19.546	7.927%
16	241.70	4.887	1.982%
32	245.37	1.222	0.495%
64	246.28	0.305	0.124%

Example 4

Use multiple-segment trapezoidal rule to find

$$I = \int_0^2 \frac{1}{\sqrt{x}} dx$$

Solution

We cannot use the trapezoidal rule for this integral, as the value of the integrand at $x = 0$ is infinite. However, it is known that a discontinuity in a curve will not change the area under it. We can assume any value for the function at $x = 0$. The algorithm to define the function so that we can use the multiple-segment trapezoidal rule is given below.

Function $f(x)$
If $x = 0$ Then $f = 0$
If $x \neq 0$ Then $f = x\wedge(-0.5)$
End Function

Basically, we are just assigning the function a value of zero at $x = 0$. Everywhere else, the function is continuous. This means the true value of our integral will be just that—true. Let's see what happens using the multiple-segment trapezoidal rule.
Using two segments, we get

$$h = \frac{2-0}{2} = 1$$

$$f(0) = 0$$

$$f(1) = \frac{1}{\sqrt{1}} = 1$$

$$f(2) = \frac{1}{\sqrt{2}} = 0.70711$$

$$I \approx \frac{b-a}{2n}\left[f(a) + 2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\} + f(b)\right]$$

$$= \frac{2-0}{2(2)}\left[f(0) + 2\left\{\sum_{i=1}^{2-1} f(0+1)\right\} + f(2)\right]$$

$$= \frac{2}{4}[f(0) + 2f(1) + f(2)]$$

$$= \frac{2}{4}[0 + 2(1) + 0.70711]$$

$$= 1.3536$$

So what is the true value of this integral?

$$\int_0^2 \frac{1}{\sqrt{x}} dx = 2.8284$$

Thus making the absolute relative true error

$$|\in_t| = \left|\frac{2.8284 - 1.3536}{2.8284}\right| \times 100$$

$$= 52.145\%$$

Table 3 Values obtained using multiple-segment trapezoidal rule for $\int_0^2 \frac{1}{\sqrt{x}} dx$.

n	Approximate Value	E_t	$\lvert\in_t\rvert$
2	1.354	1.474	52.14%
4	1.792	1.036	36.64%
8	2.097	0.731	25.85%
16	2.312	0.516	18.26%
32	2.463	0.365	12.91%
64	2.570	0.258	9.128%
128	2.646	0.182	6.454%
256	2.699	0.129	4.564%
512	2.737	0.091	3.227%
1024	2.764	0.064	2.282%
2048	2.783	0.045	1.613%
4096	2.796	0.032	1.141%