Error in Multiple-segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

Where ζ is some point in $[a,b]$.
What is the error then in the multiple-segment trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment trapezoidal rule.

The error in each segment is

$$E_1 = -\frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= -\frac{h^3}{12} f''(\zeta_1)$$

$$E_2 = -\frac{[(a+2h)-(a+h)]^3}{12} f''(\zeta_2), \quad a+h < \zeta_2 < a+2h$$

$$= -\frac{h^3}{12} f''(\zeta_2)$$

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$$E_i = -\frac{[(a+ih)-(a+(i-1)h)]^3}{12} f''(\zeta_i), \quad a+(i-1)h < \zeta_i < a+ih$$

$$= -\frac{h^3}{12} f''(\zeta_i)$$

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$$E_{n-1} = -\frac{[\{a+(n-1)h\}-\{a+(n-2)h\}]^3}{12} f''(\zeta_{n-1}), \quad a+(n-2)h < \zeta_{n-1} < a+(n-1)h$$

$$= -\frac{h^3}{12} f''(\zeta_{n-1})$$

$$E_n = -\frac{[b-\{a+(n-1)h\}]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= -\frac{h^3}{12} f''(\zeta_n)$$

Hence the total error in the multiple-segment trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i$$
$$= -\frac{h^3}{12}\sum_{i=1}^{n} f''(\zeta_i)$$
$$= -\frac{(b-a)^3}{12n^3}\sum_{i=1}^{n} f''(\zeta_i)$$
$$= -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the second derivative $f''(x),\ a < x < b$.

Hence

$$E_t = -\frac{(b-a)^3}{12n^2}\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

In Table 4, the approximate value of the integral

$$\int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt$$

is given as a function of the number of segments. You can visualize that as the number of segments are doubled, the true error gets approximately quartered.

Table 4 Values obtained using multiple-segment trapezoidal rule for

$$x = \int_8^{30}\left(2000\ln\left[\frac{140000}{140000-2100t}\right]-9.8t\right)dt.$$

n	Approximate Value	E_t	$\lvert\in_t\rvert\%$	$\lvert\in_a\rvert\%$
2	11266	-205	1.853	5.343
4	11113	-52	0.4701	0.3594
8	11074	-13	0.1175	0.03560
16	11065	-4	0.03616	0.00401

For example, for the 2-segment trapezoidal rule, the true error is -205, and a quarter of that error is -51.25. That is close to the true error of -48 for the 4-segment trapezoidal rule.

Can you answer the question w*hy is the true error not exactly -51.25?* How does this information help us in numerical integration? You will find out that this forms the basis of Romberg integration based on the trapezoidal rule, where we use the argument that true error gets approximately quartered when the number of segments is doubled. Romberg integration based on the trapezoidal rule is computationally more efficient than using the trapezoidal rule by itself in developing an automatic integration scheme.