Derivation

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

Here, we will discuss the trapezoidal rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

What is the trapezoidal rule?

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial. Integrating polynomials is simple and is based on the calculus formula.

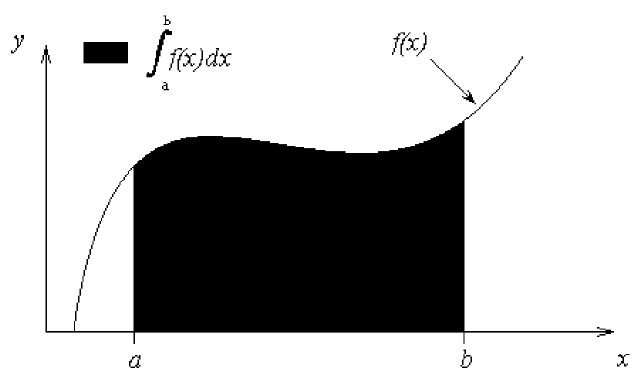

Figure 1 Integration of a function

$$\int_a^b x^n dx = \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right),\ n \neq -1 \qquad (1)$$

So if we want to approximate the integral

$$I = \int_a^b f(x)dx \qquad (2)$$

to find the value of the above integral, one assumes

$$f(x) \approx f_n(x) \qquad (3)$$

where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n . \qquad (4)$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

Derivation of the Trapezoidal Rule

Method 1: Derived from Calculus

$$\begin{aligned}\int_a^b f(x)dx &\approx \int_a^b f_1(x)dx \\ &= \int_a^b (a_0 + a_1 x)dx \\ &= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \qquad (5)\end{aligned}$$

But what is a_0 and a_1? Now if one chooses, $(a, f(a))$ and $(b, f(b))$ as the two points to approximate $f(x)$ by a straight line from a to b,

$$f(a) = f_1(a) = a_0 + a_1 a \qquad (6)$$

$$f(b) = f_1(b) = a_0 + a_1 b \qquad (7)$$

Solving the above two equations for a_1 and a_0,

$$a_1 = \frac{f(b) - f(a)}{b - a}$$

$$a_0 = \frac{f(a)b - f(b)a}{b - a} \qquad (8a)$$

Hence from Equation (5),

$$\int_a^b f(x)dx \approx \frac{f(a)b - f(b)a}{b-a}(b-a) + \frac{f(b)-f(a)}{b-a}\frac{b^2-a^2}{2} \qquad (8b)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad (9)$$

<u>Method 2: Also Derived from Calculus</u>

$f_1(x)$ can also be approximated by using Newton's divided difference polynomial as

$$f_1(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \qquad (10)$$

Hence

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

$$= \int_a^b \left[f(a) + \frac{f(b)-f(a)}{b-a}(x-a) \right] dx$$

$$= \left[f(a)x + \frac{f(b)-f(a)}{b-a}\left(\frac{x^2}{2} - ax\right) \right]_a^b$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab - \frac{a^2}{2} + a^2\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\left(\frac{b^2}{2} - ab + \frac{a^2}{2}\right)$$

$$= f(a)b - f(a)a + \left(\frac{f(b)-f(a)}{b-a}\right)\frac{1}{2}(b-a)^2$$

$$= f(a)b - f(a)a + \frac{1}{2}\left(f(b)-f(a)\right)(b-a)$$

$$= f(a)b - f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a - \frac{1}{2}f(a)b + \frac{1}{2}f(a)a$$

$$= \frac{1}{2}f(a)b - \frac{1}{2}f(a)a + \frac{1}{2}f(b)b - \frac{1}{2}f(b)a$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \qquad (11)$$

This gives the same result as Equation (10) because they are just different forms of writing the same polynomial.

Method 3: Derived from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid. The integral

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \frac{1}{2}\text{(Sum of length of parallel sides)(Perpendicular}$$

distance between parallel sides)

$$= \frac{1}{2}\left(f(b)+f(a)\right)(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{12}$$

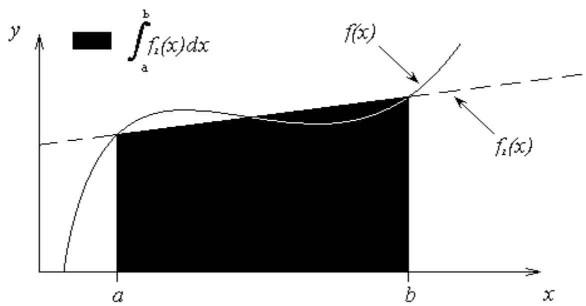

Figure 2 Geometric representation of trapezoidal rule.

Method 4: Derived from Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients. The formula

$$\int_a^b f(x)dx \approx \frac{b-a}{2}f(a)+\frac{b-a}{2}f(b) \tag{13}$$

$$= \sum_{i=1}^{2} c_i f(x_i)$$

where

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = a$$

$$x_2 = b$$

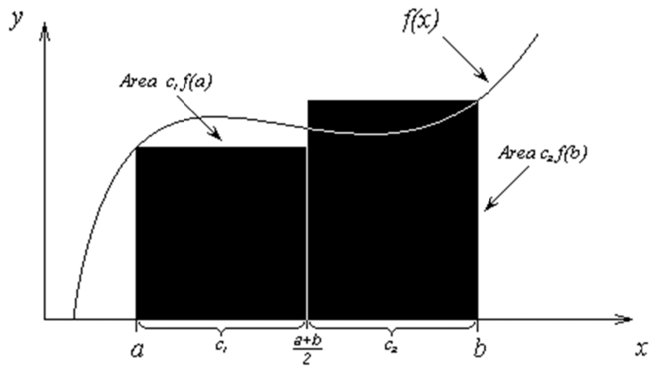

Figure 3 Area by method of coefficients.

The interpretation is that $f(x)$ is evaluated at points a and b, and each function evaluation is given a weight of $\frac{b-a}{2}$. Geometrically, Equation (12) is looked at as the area of a trapezoid, while Equation (13) is viewed as the sum of the area of two rectangles, as shown in Figure 3. How can one derive the trapezoidal rule by the method of coefficients?

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \tag{14}$$

Let the right hand side be an exact expression for integrals of $\int_a^b 1dx$ and $\int_a^b xdx$, that is, the formula will then also be exact for linear combinations of $f(x)=1$ and $f(x)=x$, that is, for $f(x)=a_0(1)+a_1(x)$.

$$\int_a^b 1dx = b-a = c_1 + c_2 \tag{15}$$

$$\int_a^b xdx = \frac{b^2-a^2}{2} = c_1 a + c_2 b \tag{16}$$

Solving the above two equations gives

$$c_1 = \frac{b-a}{2}$$
$$c_2 = \frac{b-a}{2} \tag{17}$$

Hence

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{18}$$

Method 5: Another approach on the Method of Coefficients

The trapezoidal rule can also be derived by the method of coefficients by another approach

$$\int_a^b f(x)dx \approx \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)$$

Assume

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b) \tag{19}$$

Let the right hand side be exact for integrals of the form

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left(a_0 x + a_1 \frac{x^2}{2} \right)_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2} \right) \tag{20}$$

But we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \tag{21}$$

to give the same result as Equation (20) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \tag{22}$$

Hence from Equations (20) and (22),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2} \right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary for a general straight line

$$c_1 + c_2 = b - a$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \tag{23}$$

Again, solving the above two equations (23) gives

$$
\begin{aligned}
c_1 &= \frac{b-a}{2} \\
c_2 &= \frac{b-a}{2}
\end{aligned}
\tag{24}
$$

Therefore

$$
\begin{aligned}
\int_a^b f(x)dx &\approx c_1 f(a) + c_2 f(b) \\
&= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b)
\end{aligned}
\tag{25}
$$