Example 1

For an integral $\int_{-1}^{1} f(x)dx$, show that the two-point Gauss quadrature rule approximates to

$$\int_{-1}^{1} f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$$

where

$$c_1 = 1$$
$$c_2 = 1$$
$$x_1 = -\frac{1}{\sqrt{3}}$$
$$x_2 = \frac{1}{\sqrt{3}}$$

Solution

Assuming the formula

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2) \qquad \text{(E1.1)}$$

gives exact values for integrals $\int_{-1}^{1} 1dx$, $\int_{-1}^{1} xdx$, $\int_{-1}^{1} x^2 dx$, and $\int_{-1}^{1} x^3 dx$.

Then

$$\int_{-1}^{1} 1dx = 2 = c_1 + c_2 \qquad \text{(E1.2)}$$

$$\int_{-1}^{1} xdx = 0 = c_1 x_1 + c_2 x_2 \qquad \text{(E1.3)}$$

$$\int_{-1}^{1} x^2 dx = \frac{2}{3} = c_1 x_1^{\ 2} + c_2 x_2^{\ 2} \qquad \text{(E1.4)}$$

$$\int_{-1}^{1} x^3 dx = 0 = c_1 x_1^{\ 3} + c_2 x_2^{\ 3} \qquad \text{(E1.5)}$$

Multiplying Equation (E1.3) by $x_1^{\ 2}$ and subtracting from Equation (E1.5) gives

$$c_2 x_2 \left(x_1^{\ 2} - x_2^{\ 2}\right) = 0 \qquad \text{(E1.6)}$$

The solution to the above equation is

$c_2 = 0$, or/and
$x_2 = 0$, or/and
$x_1 = x_2$, or/and
$x_1 = -x_2$.

I. $c_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 = 2,\ c_1 x_1 = 0,\ c_1 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 = 0$. But since $c_1 = 2$, then $x_1 = 0$ from $c_1 x_1 = 0$, but $x_1 = 0$ conflicts with $c_1 x_1^2 = \frac{2}{3}$.

II. $x_2 = 0$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 = 0$, $c_1 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 = 0$. Since $c_1 x_1 = 0$, then c_1 or x_1 has to be zero but this violates $c_1 x_1^2 = \frac{2}{3} \neq 0$.

III. $x_1 = x_2$ is not acceptable as Equations (E1.2-E1.5) reduce to $c_1 + c_2 = 2$, $c_1 x_1 + c_2 x_1 = 0$, $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3}$, and $c_1 x_1^3 + c_2 x_1^3 = 0$. If $x_1 \neq 0$, then $c_1 x_1 + c_2 x_1 = 0$ gives $c_1 + c_2 = 0$ and that violates $c_1 + c_2 = 2$. If $x_1 = 0$, then that violates $c_1 x_1^2 + c_2 x_1^2 = \frac{2}{3} \neq 0$.

That leaves the solution of $x_1 = -x_2$ as the only possible acceptable solution and in fact, it does not have violations (see it for yourself)

$$x_1 = -x_2 \qquad \text{(E1.7)}$$

Substituting (E1.7) in Equation (E1.3) gives

$$c_1 = c_2 \qquad \text{(E1.8)}$$

From Equations (E1.2) and (E1.8),

$$c_1 = c_2 = 1 \qquad \text{(E1.9)}$$

Equations (E1.4) and (E1.9) gives

$$x_1^2 + x_2^2 = \frac{2}{3} \qquad \text{(E1.10)}$$

Since Equation (E1.7) requires that the two results be of opposite sign, we get

$$x_1 = -\frac{1}{\sqrt{3}}$$

$$x_2 = \frac{1}{\sqrt{3}}$$

Hence

$$\int_{-1}^{1} f(x)dx = c_1 f(x_1) + c_2 f(x_2)$$

(E1.11)

$$= f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

Example 2

For an integral $\int_a^b f(x)dx,$ derive the one-point Gauss quadrature rule.

Solution

The one-point Gauss quadrature rule is

$$\int_a^b f(x)dx \approx c_1 f(x_1) \qquad \text{(E2.1)}$$

Assuming the formula gives exact values for integrals $\int_{-1}^{1} 1dx,$ and $\int_{-1}^{1} xdx$

$$\int_a^b 1dx = b - a = c_1$$

$$\int_a^b xdx = \frac{b^2 - a^2}{2} = c_1 x_1 \qquad \text{(E2.2)}$$

Since $c_1 = b - a,$ the other equation becomes

$$(b-a)x_1 = \frac{b^2 - a^2}{2}$$

$$x_1 = \frac{b+a}{2} \qquad \text{(E2.3)}$$

Therefore, one-point Gauss quadrature rule can be expressed as

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{b+a}{2}\right) \qquad \text{(E2.4)}$$

Example 3

What would be the formula for

$$\int_a^b f(x)dx = c_1 f(a) + c_2 f(b)$$

if you want the above formula to give you exact values of $\int_a^b \left(a_0 x + b_0 x^2\right)dx$, that is, a linear combination of x and x^2.

Solution

If the formula is exact for a linear combination of x and x^2, then

$$\int_a^b x dx = \frac{b^2 - a^2}{2} = c_1 a + c_2 b$$

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3} = c_1 a^2 + c_2 b^2 \qquad \text{(E3.1)}$$

Solving the two Equations (E3.1) simultaneously gives

$$\begin{bmatrix} a & b \\ a^2 & b^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \dfrac{b^2 - a^2}{2} \\ \dfrac{b^3 - a^3}{3} \end{bmatrix}$$

$$c_1 = -\frac{1}{6}\frac{-ab - b^2 + 2a^2}{a}$$

$$c_2 = -\frac{1}{6}\frac{a^2 + ab - 2b^2}{b} \qquad \text{(E3.2)}$$

So

$$\int_a^b f(x)dx = -\frac{1}{6}\frac{-ab - b^2 + 2a^2}{a} f(a) - \frac{1}{6}\frac{a^2 + ab - 2b^2}{b} f(b)$$

(E3.3)

Let us see if the formula works.

Evaluate $\int_2^5 \left(2x^2 - 3x\right)dx$ using Equation(E3.3)

$$\int_2^5 \left(2x^2 - 3x\right)dx \approx c_1 f(a) + c_2 f(b)$$

$$= -\frac{1}{6}\frac{-(2)(5) - 5^2 + 2(2)^2}{2}\left[2(2)^2 - 3(2)\right] - \frac{1}{6}\frac{2^2 + 2(5) - 2(5)^2}{5}[2(5)^2 - 3(5)]$$

$$= 46.5$$

The exact value of $\int_2^5 (2x^2 - 3x)dx$ is given by

$$\int_2^5 (2x^2 - 3x)dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2}\right]_2^5$$
$$= 46.5$$

Any surprises?

Now evaluate $\int_2^5 3dx$ using Equation (E3.3)

$$\int_2^5 3dx \approx c_1 f(a) + c_2 f(b)$$
$$= -\frac{1}{6}\frac{-2(5) - 5^2 + 2(2)^2}{2}(3) - \frac{1}{6}\frac{2^2 + 2(5) - 2(5)^2}{5}(3)$$
$$= 10.35$$

The exact value of $\int_2^5 3dx$ is given by

$$\int_2^5 3dx = [3x]_2^5$$
$$= 9$$

Because the formula will only give exact values for linear combinations of x and x^2, it does not work exactly even for a simple integral of $\int_2^5 3dx$.

Do you see now why we choose $a_0 + a_1 x$ as the integrand for which the formula

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

gives us exact values?

Example 4

Use two-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_{8}^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation(23) gives

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2} \int_{-1}^{1} f\left(\frac{30-8}{2}x + \frac{30+8}{2} \right) dx$$

$$= 11 \int_{-1}^{1} f(11x + 19) dx$$

Next, get weighting factors and function argument values from Table 1 for the two point rule,

$c_1 = \ 1.000000000\,.$

$x_1 = -0.577350269$

$c_2 = \ 1.000000000$

$x_2 = \ 0.577350269$

Now we can use the Gauss quadrature formula

$$11 \int_{-1}^{1} f(11x + 19) dx \approx 11 \left[c_1 f(11x_1 + 19) + c_2 f(11x_2 + 19) \right]$$

$$= 11\left[f(11(-0.5773503) + 19) + f(11(0.5773503) + 19) \right]$$

$$= 11\left[f(12.64915) + f(25.35085) \right]$$

$$= 11\left[(296.8317) + (708.4811) \right]$$

$$= 11058.44 \text{ m}$$

since

$$f(12.64915) = 2000 \ln\left[\frac{140000}{140000 - 2100(12.64915)} \right] - 9.8(12.64915)$$

$$= 296.8317$$

$$f(25.35085) = 2000 \ln\left[\frac{140000}{140000 - 2100(25.35085)} \right] - 9.8(25.35085)$$

$$= 708.4811$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left| \frac{11061.34 - 11058.44}{11061.34} \right| \times 100$$

$$= 0.0262\%$$

Example 5

Use three-point Gauss quadrature rule to approximate the distance covered by a rocket from $t = 8$ to $t = 30$ as given by

$$x = \int_{8}^{30} \left(2000 \ln\left[\frac{140000}{140000 - 2100t} \right] - 9.8t \right) dt$$

Also, find the absolute relative true error.

Solution

First, change the limits of integration from $[8, 30]$ to $[-1, 1]$ using Equation (23) gives

$$\int_{8}^{30} f(t)dt = \frac{30-8}{2} \int_{-1}^{1} f\left(\frac{30-8}{2} x + \frac{30+8}{2} \right) dx$$

$$= 11 \int_{-1}^{1} f(11x + 19) dx$$

The weighting factors and function argument values are

$c_1 = \ \ 0.555555556$

$x_1 = -0.774596669$

$c_2 = \ \ 0.888888889$

$x_2 = \ \ 0.000000000$

$c_3 = \ \ 0.555555556$

$x_3 = \ \ 0.774596669$

and the formula is

$$11 \int_{-1}^{1} f(11x + 19) dx \approx 11\left[c_1 f(11x_1 + 19) + c_2 f(11x_2 + 19) + c_3 f(11x_3 + 19) \right]$$

$$= 11 \begin{bmatrix} 0.5555556 f(11(-.7745967) + 19) + 0.8888889 f(11(0.0000000) + 19) \\ + 0.5555556 f(11(0.7745967) + 19) \end{bmatrix}$$

$$= 11[0.55556 f(10.47944) + 0.88889 f(19.00000) + 0.55556 f(27.52056)]$$

$$= 11[0.55556 \times 239.3327 + 0.88889 \times 484.7455 + 0.55556 \times 795.1069]$$

$$= 11061.31 \text{ m}$$

since

$$f(10.47944) = 2000 \ln\left[\frac{140000}{140000 - 2100(10.47944)} \right] - 9.8(10.47944)$$

$$= 239.3327$$

$$f(19.00000) = 2000 \ln\left[\frac{140000}{140000 - 2100(19.00000)} \right] - 9.8(19.00000)$$

$$= 484.7455$$

$$f(27.52056) = 2000\ln\left[\frac{140000}{140000 - 2100(27.52056)}\right] - 9.8(27.52056)$$

$$= 795.1069$$

The absolute relative true error, $|\in_t|$, is (True value = 11061.34 m)

$$|\in_t| = \left|\frac{11061.34 - 11061.31}{11061.34}\right| \times 100$$

$$= 0.0003\%$$