Background

To derive the trapezoidal rule from the method of undetermined coefficients, we approximated

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b) \qquad (1)$$

Let the right hand side be exact for integrals of a straight line, that is, for an integrated form of

$$\int_a^b (a_0 + a_1 x)dx$$

So

$$\int_a^b (a_0 + a_1 x)dx = \left[a_0 x + a_1 \frac{x^2}{2} \right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) \qquad (2)$$

But from Equation (1), we want

$$\int_a^b (a_0 + a_1 x)dx = c_1 f(a) + c_2 f(b) \qquad (3)$$

to give the same result as Equation (2) for $f(x) = a_0 + a_1 x$.

$$\int_a^b (a_0 + a_1 x)dx = c_1(a_0 + a_1 a) + c_2(a_0 + a_1 b)$$

$$= a_0(c_1 + c_2) + a_1(c_1 a + c_2 b) \qquad (4)$$

Hence from Equations (2) and (4),

$$a_0(b-a) + a_1\left(\frac{b^2 - a^2}{2}\right) = a_0(c_1 + c_2) + a_1(c_1 a + c_2 b)$$

Since a_0 and a_1 are arbitrary constants for a general straight line

$$c_1 + c_2 = b - a \qquad (5a)$$

$$c_1 a + c_2 b = \frac{b^2 - a^2}{2} \qquad (5b)$$

Multiplying Equation (5a) by a and subtracting from Equation (5b) gives

$$c_2 = \frac{b-a}{2} \tag{6a}$$

Substituting the above found value of c_2 in Equation (5a) gives

$$c_1 = \frac{b-a}{2} \tag{6b}$$

Therefore

$$\int_a^b f(x)dx \approx c_1 f(a) + c_2 f(b)$$

$$= \frac{b-a}{2} f(a) + \frac{b-a}{2} f(b) \tag{7}$$