Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Table 1 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

Determine the distance, s, covered by the rocket from $t = 11$ to $t = 16$ using the velocity data provided and use any applicable numerical technique.

Solution

Method 1: Average Velocity Method

The velocity of the rocket is not provided at $t = 11$ and $t = 16$, so we will have to use an interval that includes $[11, 16]$ to find the average velocity of the rocket within that range. In this case, the interval $[10, 20]$ will suffice.

$$v(10) = 227.04$$
$$v(15) = 362.78$$
$$v(20) = 517.35$$

$$\begin{aligned} Average\ Velocity &= \frac{v(10) + v(15) + v(20)}{3} \\ &= \frac{227.04 + 362.78 + 517.35}{3} \\ &= 369.06 \text{ m/s} \end{aligned}$$

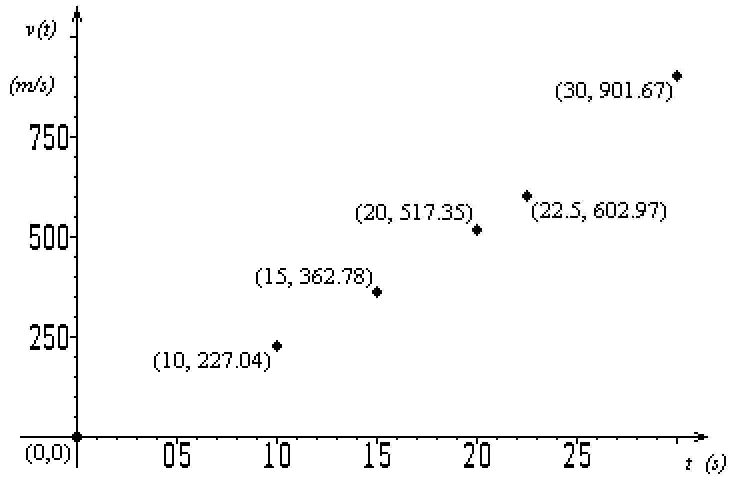

Figure 1 Velocity vs. time data for the rocket example

Using

$$s = \bar{v}\Delta t,$$

we get

$$s = (369.06)(16-11) = 1845.3 \text{ m}$$

<u>Method 2: Trapezoidal Rule</u>

If we were finding the distance traveled between times in the data table, we would simply find the area of the trapezoids with the corner points given as the velocity and time data points. For example

$$\int_{10}^{20} v(t)dt = \int_{10}^{15} v(t)dt + \int_{15}^{20} v(t)dt$$

and applying the trapezoidal rule over each of the above integrals gives

$$\int_{10}^{20} v(t)dt \quad \approx \frac{15-10}{2}[v(10)+v(15)] + \frac{20-15}{2}[v(15)+v(20)]$$

The values of $v(10)$, $v(15)$ and $v(20)$ are given in Table 1.

However, we are interested in finding

$$\int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt$$

and applying the trapezoidal rule over each of the above integrals gives

$$\int_{11}^{16} v(t)dt \approx \frac{15-11}{2}[v(11)+v(15)] + \frac{16-15}{2}[v(15)+v(16)]$$

$$= \frac{15-11}{2}(v(11)+362.78) + \frac{16-15}{2}(362.78+v(16))$$

How do we find $v(11)$ and $v(16)$? We use linear interpolation. To find $v(11)$,

$$v(t) = 227.04 + 27.148(t - 10), \ 10 \le t \le 15$$
$$v(11) = 227.04 + 27.148(11 - 10)$$
$$= 254.19 \text{ m/s}$$

and to find $v(16)$

$$v(t) = 362.78 + 30.913(t - 15), \ 15 \le t \le 20$$
$$v(16) = 362.78 + 30.913(16 - 15)$$
$$= 393.69 \text{ m/s}$$

Then

$$\int_{11}^{16} v(t)dt \approx \frac{15-11}{2}(v(11) + 362.78) + \frac{16-15}{2}(362.78 + v(16))$$

$$= \frac{15-11}{2}(254.19 + 362.78) + \frac{16-15}{2}(362.78 + 393.69)$$
$$= 1612.2 \text{ m}$$

<u>Method 3: Polynomial interpolation to find the velocity profile</u>

Because we are finding the area under the curve from $[10, 20]$, we must use three points, $t = 10$, $t = 15$, and $t = 20$, to fit a quadratic polynomial through the data. Using polynomial interpolation, our resulting velocity function is (refer to notes on direct method of interpolation)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \ 10 \le t \le 20.$$

Now, we simply take the integral of the quadratic within our limits, giving us

$$s \approx \int_{11}^{16} \left(12.05 + 17.733t + 0.3766t^2\right)dt$$
$$= \left[12.05t + \frac{17.733t^2}{2} + \frac{0.3766t^3}{3}\right]_{11}^{16}$$
$$= 12.05(16 - 11) + \frac{17.733}{2}\left(16^2 - 11^2\right) + \frac{0.3766}{3}\left(16^3 - 11^3\right)$$
$$= 1604.3 \text{ m}$$

Method 4: Spline interpolation to find the velocity profile

Fitting quadratic splines (refer to notes on spline method of interpolation) through the data results in the following set of quadratics

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$

The value of the integral would then simply be

$$s = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt$$

$$\approx \int_{11}^{15} \left(0.8888t^2 + 4.928t + 88.88\right)dt + \int_{15}^{16} \left(-0.1356t^2 + 35.66t - 141.61\right)dt$$

$$= \left[\frac{0.8888t^3}{3} + \frac{4.928t^2}{2} + 88.88t\right]_{11}^{15} + \left[\frac{-0.1356t^3}{3} + \frac{35.66t^2}{2} - 141.61t\right]_{15}^{16}$$

$$= \frac{0.8888}{3}\left(15^3 - 11^3\right) + \frac{4.928}{2}\left(15^2 - 11^2\right) + 88.88(15 - 11)$$

$$+ \frac{-0.1356}{3}\left(16^3 - 15^3\right) + \frac{35.66}{2}\left(16^2 - 15^2\right) - 141.61(16 - 15)$$

$$= 1595.9 \ m$$