Quadratic Spline Interpolation

In these splines, a quadratic polynomial approximates the data between two consecutive data points. Given $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, fit quadratic splines through the data. The splines are given by

$$
\begin{aligned}
f(x) &= a_1 x^2 + b_1 x + c_1, & x_0 \le x \le x_1 \\
&= a_2 x^2 + b_2 x + c_2, & x_1 \le x \le x_2 \\
&. \\
&. \\
&. \\
&= a_n x^2 + b_n x + c_n, & x_{n-1} \le x \le x_n
\end{aligned}
$$

So how does one find the coefficients of these quadratic splines? There are $3n$ such coefficients

$$a_i,\ i = 1,2,\ldots\ldots,n$$
$$b_i,\ i = 1,2,\ldots\ldots,n$$
$$c_i,\ i = 1,2,\ldots\ldots,n$$

To find $3n$ unknowns, one needs to set up $3n$ equations and then simultaneously solve them. These $3n$ equations are found as follows.

1. Each quadratic spline goes through two consecutive data points

$$
\begin{aligned}
&a_1 x_0^{\,2} + b_1 x_0 + c_1 = f(x_0) \\
&a_1 x_1^{\,2} + b_1 x_1 + c_1 = f(x_1) \\
&. \\
&. \\
&. \\
&a_i x_{i-1}^{\,2} + b_i x_{i-1} + c_i = f(x_{i-1}) \\
&a_i x_i^{\,2} + b_i x_i + c_i = f(x_i) \\
&. \\
&. \\
&. \\
&a_n x_{n-1}^{\,2} + b_n x_{n-1} + c_n = f(x_{n-1}) \\
&a_n x_n^{\,2} + b_n x_n + c_n = f(x_n)
\end{aligned}
$$

This condition gives $2n$ equations as there are n quadratic splines going through two consecutive data points.

2. The first derivatives of two quadratic splines are continuous at the interior points. For example, the derivative of the first spline

$$a_1x^2 + b_1x + c_1$$

is

$$2a_1x + b_1$$

The derivative of the second spline

$$a_2x^2 + b_2x + c_2$$

is

$$2a_2x + b_2$$

and the two are equal at $x = x_1$ giving

$$2a_1x_1 + b_1 = 2a_2x_1 + b_2$$

$$2a_1x_1 + b_1 - 2a_2x_1 - b_2 = 0$$

Similarly at the other interior points,

$$2a_2x_2 + b_2 - 2a_3x_2 - b_3 = 0$$

$$\vdots$$

$$2a_ix_i + b_i - 2a_{i+1}x_i - b_{i+1} = 0$$

$$\vdots$$

$$2a_{n-1}x_{n-1} + b_{n-1} - 2a_nx_{n-1} - b_n = 0$$

Since there are $(n-1)$ interior points, we have $(n-1)$ such equations. So far, the total number of equations is $(2n)+(n-1)=(3n-1)$ equations. We still then need one more equation.

We can assume that the first spline is linear, that is

$$a_1 = 0$$

This gives us $3n$ equations and $3n$ unknowns. These can be solved by a number of techniques used to solve simultaneous linear equations.