Example 2

The upward velocity of a rocket is given as a function of time as

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using quadratic splines.
b) Using the quadratic splines as velocity functions, find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.
c) Using the quadratic splines as velocity functions, find the acceleration of the rocket at $t = 16\text{s}$.

Solution

a) Since there are six data points, five quadratic splines pass through them.

$$
\begin{aligned}
v(t) &= a_1 t^2 + b_1 t + c_1, \quad 0 \le t \le 10 \\
&= a_2 t^2 + b_2 t + c_2, \quad 10 \le t \le 15 \\
&= a_3 t^2 + b_3 t + c_3, \quad 15 \le t \le 20 \\
&= a_4 t^2 + b_4 t + c_4, \quad 20 \le t \le 22.5 \\
&= a_5 t^2 + b_5 t + c_5, \quad 22.5 \le t \le 30
\end{aligned}
$$

The equations are found as follows.

1. Each quadratic spline passes through two consecutive data points.

$a_1 t^2 + b_1 t + c_1$ passes through $t = 0$ and $t = 10$.

$$a_1 (0)^2 + b_1 (0) + c_1 = 0 \qquad (1)$$
$$a_1 (10)^2 + b_1 (10) + c_1 = 227.04 \qquad (2)$$

$a_2 t^2 + b_2 t + c_2$ passes through $t = 10$ and $t = 15$.

$$a_2 (10)^2 + b_2 (10) + c_2 = 227.04 \qquad (3)$$
$$a_2 (15)^2 + b_2 (15) + c_2 = 362.78 \qquad (4)$$

$a_3 t^2 + b_3 t + c_3$ passes through $t = 15$ and $t = 20$.

$$a_3 (15)^2 + b_3 (15) + c_3 = 362.78 \qquad (5)$$
$$a_3 (20)^2 + b_3 (20) + c_3 = 517.35 \qquad (6)$$

$a_4t^2 + b_4t + c_4$ passes through $t = 20$ and $t = 22.5$.

$$a_4(20)^2 + b_4(20) + c_4 = 517.35 \qquad (7)$$

$$a_4(22.5)^2 + b_4(22.5) + c_4 = 602.97 \qquad (8)$$

$a_5t^2 + b_5t + c_5$ passes through $t = 22.5$ and $t = 30$.

$$a_5(22.5)^2 + b_5(22.5) + c_5 = 602.97 \qquad (9)$$

$$a_5(30)^2 + b_5(30) + c_5 = 901.67 \qquad (10)$$

2. Quadratic splines have continuous derivatives at the interior data points.

At $t = 10$

$$2a_1(10) + b_1 - 2a_2(10) - b_2 = 0 \qquad (11)$$

At $t = 15$

$$2a_2(15) + b_2 - 2a_3(15) - b_3 = 0 \qquad (12)$$

At $t = 20$

$$2a_3(20) + b_3 - 2a_4(20) - b_4 = 0 \qquad (13)$$

At $t = 22.5$

$$2a_4(22.5) + b_4 - 2a_5(22.5) - b_5 = 0 \qquad (14)$$

3. Assuming the first spline $a_1t^2 + b_1t + c_1$ is linear,

$$a_1 = 0 \qquad (15)$$

Combining Equation (1) – (15) in matrix form gives

$$\begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 100 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 225 & 15 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 400 & 20 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 506.25 & 22.5 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 900 & 30 & 1 \\
20 & 1 & 0 & -20 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 30 & 1 & 0 & -30 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 40 & 1 & 0 & -40 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 45 & 1 & 0 & -45 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ a_2 \\ b_2 \\ c_2 \\ a_3 \\ b_3 \\ c_3 \\ a_4 \\ b_4 \\ c_4 \\ a_5 \\ b_5 \\ c_5 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 227.04 \\ 227.04 \\ 362.78 \\ 362.78 \\ 517.35 \\ 517.35 \\ 602.97 \\ 602.97 \\ 901.67 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the above 15 equations give the 15 unknowns as

i	a_i	b_i	c_i
1	0	22.704	0
2	0.8888	4.928	88.88
3	–0.1356	35.66	–141.61
4	1.6048	–33.956	554.55
5	0.20889	28.86	–152.13

Therefore, the splines are given by

$$
\begin{aligned}
v(t) &= 22.704t, && 0 \le t \le 10 \\
&= 0.8888t^2 + 4.928t + 88.88, && 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, && 15 \le t \le 20 \\
&= 1.6048t^2 - 33.956t + 554.55, && 20 \le t \le 22.5 \\
&= 0.20889t^2 + 28.86t - 152.13, && 22.5 \le t \le 30
\end{aligned}
$$

At $t = 16\text{s}$

$$
\begin{aligned}
v(16) &= -0.1356(16)^2 + 35.66(16) - 141.61 \\
&= 394.24\,\text{m/s}
\end{aligned}
$$

b) The distance covered by the rocket between 11 and 16 seconds can be calculated as

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$

But since the splines are valid over different ranges, we need to break the integral accordingly as

$$
\begin{aligned}
v(t) &= 0.8888t^2 + 4.928t + 88.88, \quad 10 \le t \le 15 \\
&= -0.1356t^2 + 35.66t - 141.61, \quad 15 \le t \le 20
\end{aligned}
$$

$$\int_{11}^{16} v(t)dt = \int_{11}^{15} v(t)dt + \int_{15}^{16} v(t)dt$$

$$s(16) - s(11) = \int_{11}^{15} (0.8888t^2 + 4.928t + 88.88)dt + \int_{15}^{16} (-0.1356t^2 + 35.66t - 141.61)dt$$

$$
\begin{aligned}
&= \left[0.8888\frac{t^3}{3} + 4.928\frac{t^2}{2} + 88.88t\right]_{11}^{15} \\
&\quad + \left[-0.1356\frac{t^3}{3} + 35.66\frac{t^2}{2} - 141.61t\right]_{15}^{16} \\
&= 1217.35 + 378.53 \\
&= 1595.9 \text{ m}
\end{aligned}
$$

c) What is the acceleration at $t = 16$?

$$a(16) = \frac{d}{dt} v(t)\Big|_{t=16}$$

$$a(t) = \frac{d}{dt} v(t) = \frac{d}{dt}(-0.1356t^2 + 35.66t - 141.61)$$

$$= -0.2712t + 35.66, \qquad 15 \le t \le 20$$

$$a(16) = -0.2712(16) + 35.66$$

$$= 31.321 \text{m/s}^2$$