Direct Method - Cubic

Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Table 3 Velocity as a function of time.

t (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

a) Determine the value of the velocity at $t = 16$ seconds using the direct method of interpolation and a third order polynomial.
b) Find the absolute relative approximate error for the third order polynomial approximation.
c) Using the third order polynomial interpolant for velocity from part (a), find the distance covered by the rocket from $t = 11\text{s}$ to $t = 16\text{s}$.
d) Using the third order polynomial interpolant for velocity from part (a), find the acceleration of the rocket at $t = 16\text{s}$.

Solution

a) For third order polynomial interpolation (also called cubic interpolation), we choose the velocity given by

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

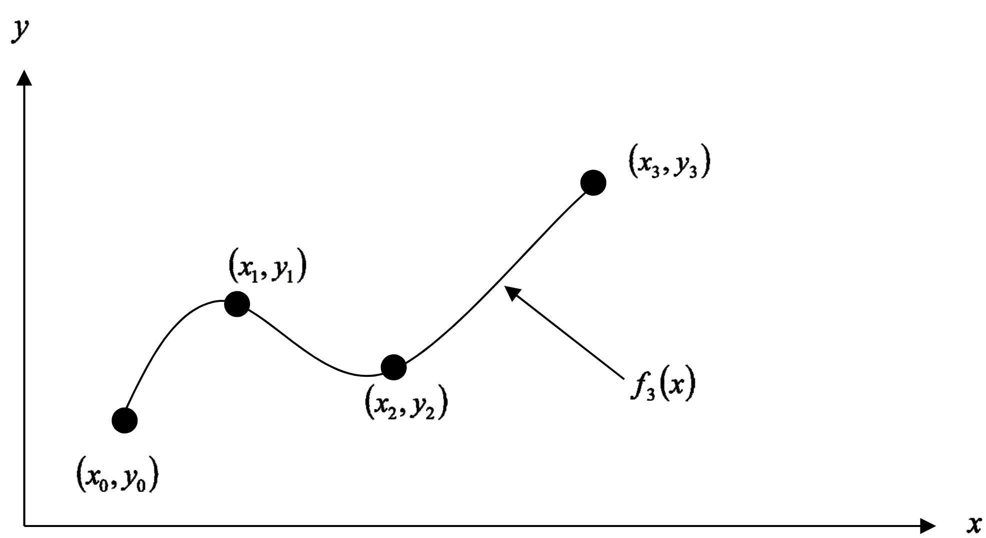

Figure 5 Cubic interpolation.

Since we want to find the velocity at $t = 16$, and we are using a third order polynomial, we need to choose the four data points closest to $t = 16$ that also bracket $t = 16$ to evaluate it.

The four points are $t_0 = 10,\ t_1 = 15,\ t_2 = 20$ and $t_3 = 22.5$.

Then

$$\begin{aligned} t_0 &= 10, & v(t_0) &= 227.04 \\ t_1 &= 15, & v(t_1) &= 362.78 \\ t_2 &= 20, & v(t_2) &= 517.35 \\ t_3 &= 22.5, & v(t_3) &= 602.97 \end{aligned}$$

gives

$$\begin{aligned} v(10) &= a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3 = 227.04 \\ v(15) &= a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3 = 362.78 \\ v(20) &= a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3 = 517.35 \\ v(22.5) &= a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3 = 602.97 \end{aligned}$$

Writing the four equations in matrix form, we have

$$\begin{bmatrix} 1 & 10 & 100 & 1000 \\ 1 & 15 & 225 & 3375 \\ 1 & 20 & 400 & 8000 \\ 1 & 22.5 & 506.25 & 11391 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \\ 602.97 \end{bmatrix}$$

Solving the above four equations gives

$$\begin{aligned} a_0 &= -4.2540 \\ a_1 &= 21.266 \\ a_2 &= 0.13204 \\ a_3 &= 0.0054347 \end{aligned}$$

Hence

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$= -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$

$$\begin{aligned} v(16) &= -4.2540 + 21.266(16) + 0.13204(16)^2 + 0.0054347(16)^3 \\ &= 392.06\,\text{m/s} \end{aligned}$$

b) The absolute percentage relative approximate error $|\in_a|$ for the value obtained for $v(16)$ between second and third order polynomial is

$$\begin{aligned} |\in_a| &= \left|\frac{392.06 - 392.19}{392.06}\right| \times 100 \\ &= 0.033269\% \end{aligned}$$

c) The distance covered by the rocket between $t = 11\text{s}$ and $t = 16\text{s}$ can be calculated from the interpolating polynomial

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$

Note that the polynomial is valid between $t = 10$ and $t = 22.5$ and hence includes the limits of integration of $t = 11$ and $t = 16$.
So

$$s(16) - s(11) = \int_{11}^{16} v(t)dt$$

$$= \int_{11}^{16} (-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3)\, dt$$

$$=$$

$$\left[-4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4}\right]_{11}^{16}$$

$$= 1605 \text{ m}$$

d) The acceleration at $t = 16$ is given by

$$a(16) = \frac{d}{dt}v(t)\Big|_{t=16}$$

Given that

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \le t \le 22.5$$

$$\begin{aligned} a(t) &= \frac{d}{dt}v(t) \\ &= \frac{d}{dt}\left(-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3\right) \\ &= 21.266 + 0.26408t + 0.016304t^2, \quad 10 \le t \le 22.5 \end{aligned}$$

$$\begin{aligned} a(16) &= 21.266 + 0.26408(16) + 0.016304(16)^2 \\ &= 29.665 \text{m/s}^2 \end{aligned}$$