Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.

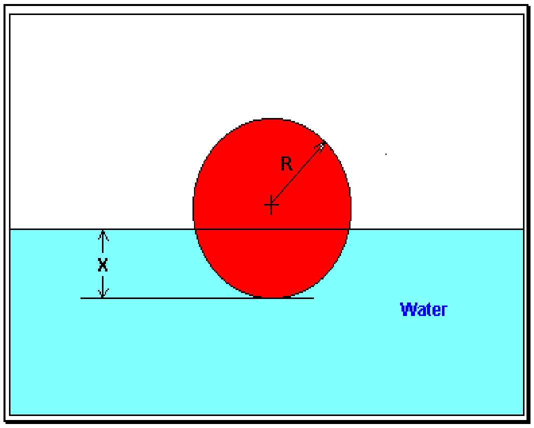

Figure 2 Floating ball problem.

The equation that gives the depth x in meters to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993\times10^{-4} = 0$$

Use the Newton-Raphson method of finding roots of equations to find

a) the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation.
b) the absolute relative approximate error at the end of each iteration, and
c) the number of significant digits at least correct at the end of each iteration.

Solution

$$f(x) = x^3 - 0.165x^2 + 3.993\times10^{-4}$$
$$f'(x) = 3x^2 - 0.33x$$

Let us assume the initial guess of the root of $f(x)=0$ is $x_0 = 0.05\,\text{m}$. This is a reasonable guess (discuss why $x=0$ and $x=0.11\,\text{m}$ are not good choices) as the extreme values of the depth x would be 0 and the diameter (0.11 m) of the ball.

<u>Iteration 1</u>

The estimate of the root is

$$\begin{aligned}
x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\
&= 0.05 - \frac{(0.05)^3 - 0.165(0.05)^2 + 3.993\times10^{-4}}{3(0.05)^2 - 0.33(0.05)} \\
&= 0.05 - \frac{1.118\times10^{-4}}{-9\times10^{-3}} \\
&= 0.05 - (-0.01242) \\
&= 0.06242
\end{aligned}$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 1 is

$$\begin{aligned}
|\in_a| &= \left|\frac{x_1 - x_0}{x_1}\right|\times100 \\
&= \left|\frac{0.06242 - 0.05}{0.06242}\right|\times100 \\
&= 19.90\%
\end{aligned}$$

The number of significant digits at least correct is 0, as you need an absolute relative approximate error of 5% or less for at least one significant digit to be correct in your result.

Iteration 2

The estimate of the root is

$$
\begin{aligned}
x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\
&= 0.06242 - \frac{(0.06242)^3 - 0.165(0.06242)^2 + 3.993\times10^{-4}}{3(0.06242)^2 - 0.33(0.06242)} \\
&= 0.06242 - \frac{-3.97781\times10^{-7}}{-8.90973\times10^{-3}} \\
&= 0.06242 - \left(4.4646\times10^{-5}\right) \\
&= 0.06238
\end{aligned}
$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$
\begin{aligned}
|\in_a| &= \left|\frac{x_2 - x_1}{x_2}\right|\times100 \\
&= \left|\frac{0.06238 - 0.06242}{0.06238}\right|\times100 \\
&= 0.0716\%
\end{aligned}
$$

The maximum value of m for which $|\in_a| \le 0.5\times10^{2-m}$ is 2.844. Hence, the number of significant digits at least correct in the answer is 2.

Iteration 3

The estimate of the root is

$$
\begin{aligned}
x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\
&= 0.06238 - \frac{(0.06238)^3 - 0.165(0.06238)^2 + 3.993\times10^{-4}}{3(0.06238)^2 - 0.33(0.06238)} \\
&= 0.06238 - \frac{4.44\times10^{-11}}{-8.91171\times10^{-3}} \\
&= 0.06238 - \left(-4.9822\times10^{-9}\right) \\
&= 0.06238
\end{aligned}
$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 3 is

$$
\begin{aligned}
|\in_a| &= \left|\frac{0.06238 - 0.06238}{0.06238}\right|\times100 \\
&= 0
\end{aligned}
$$

The number of significant digits at least correct is 4, as only 4 significant digits are carried through in all the calculations.