Example 1

You are working for 'DOWN THE TOILET COMPANY' that makes floats for ABC commodes. The floating ball has a specific gravity of 0.6 and has a radius of 5.5 cm. You are asked to find the depth to which the ball is submerged when floating in water.
The equation that gives the depth x to which the ball is submerged under water is given by

$$x^3 - 0.165x^2 + 3.993 \times 10^{-4} = 0$$

Use the bisection method of finding roots of equations to find the depth x to which the ball is submerged under water. Conduct three iterations to estimate the root of the above equation. Find the absolute relative approximate error at the end of each iteration, and the number of significant digits at least correct at the end of each iteration.

Solution

From the physics of the problem, the ball would be submerged between $x = 0$ and $x = 2R$,
where

$$R = \text{radius of the ball},$$

that is

$$0 \le x \le 2R$$
$$0 \le x \le 2(0.055)$$
$$0 \le x \le 0.11$$

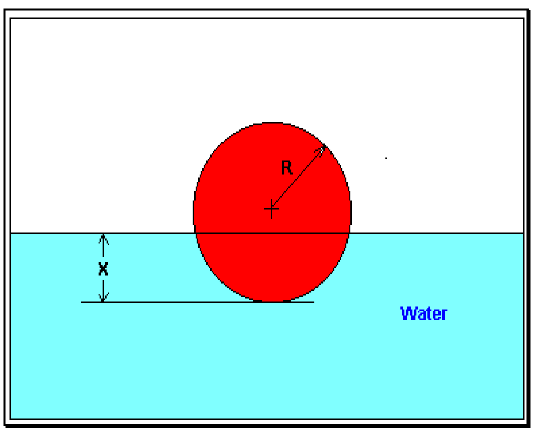

Figure 5 Floating ball problem.

Let us assume

$$x_\ell = 0, x_u = 0.11$$

Check if the function changes sign between x_ℓ and x_u.

$$f(x_\ell) = f(0) = (0)^3 - 0.165(0)^2 + 3.993 \times 10^{-4} = 3.993 \times 10^{-4}$$
$$f(x_u) = f(0.11) = (0.11)^3 - 0.165(0.11)^2 + 3.993 \times 10^{-4} = -2.662 \times 10^{-4}$$

Hence

$$f(x_\ell)f(x_u) = f(0)f(0.11) = (3.993 \times 10^{-4})(-2.662 \times 10^{-4}) < 0$$

So there is at least one root between x_ℓ and x_u, that is between 0 and 0.11.

Iteration 1

The estimate of the root is

$$
\begin{aligned}
x_m &= \frac{x_\ell + x_u}{2} \\
&= \frac{0 + 0.11}{2} \\
&= 0.055
\end{aligned}
$$

$$f(x_m) = f(0.055) = (0.055)^3 - 0.165(0.055)^2 + 3.993\times10^{-4} = 6.655\times10^{-5}$$

$$f(x_\ell)f(x_m) = f(0)f(0.055) = \left(3.993\times10^{-4}\right)\left(6.655\times10^{-4}\right) > 0$$

Hence the root is bracketed between x_m and x_u, that is, between 0.055 and 0.11. So, the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.11$$

At this point, the absolute relative approximate error $|\in_a|$ cannot be calculated as we do not have a previous approximation.

Iteration 2

The estimate of the root is

$$
\begin{aligned}
x_m &= \frac{x_\ell + x_u}{2} \\
&= \frac{0.055 + 0.11}{2} \\
&= 0.0825
\end{aligned}
$$

$$f(x_m) = f(0.0825) = (0.0825)^3 - 0.165(0.0825)^2 + 3.993\times10^{-4} = -1.622\times10^{-4}$$

$$f(x_\ell)f(x_m) = f(0.055)f(0.0825) = \left(6.655\times10^{-5}\right)\times\left(-1.622\times10^{-4}\right) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.0825. So the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.0825$$

The absolute relative approximate error $|\in_a|$ at the end of Iteration 2 is

$$
\begin{aligned}
|\in_a| &= \left|\frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}}\right| \times 100 \\
&= \left|\frac{0.0825 - 0.055}{0.0825}\right| \times 100 \\
&= 33.33\%
\end{aligned}
$$

None of the significant digits are at least correct in the estimated root of $x_m = 0.0825$ because the absolute relative approximate error is greater than 5%.

Iteration 3

$$x_m = \frac{x_\ell + x_u}{2}$$
$$= \frac{0.055 + 0.0825}{2}$$
$$= 0.06875$$
$$f(x_m) = f(0.06875) = (0.06875)^3 - 0.165(0.06875)^2 + 3.993 \times 10^{-4} = -5.563 \times 10^{-5}$$

$$f(x_\ell) f(x_m) = f(0.055) f(0.06875) = (6.655 \times 10^5) \times (-5.563 \times 10^{-5}) < 0$$

Hence, the root is bracketed between x_ℓ and x_m, that is, between 0.055 and 0.06875. So the lower and upper limit of the new bracket is

$$x_\ell = 0.055,\ x_u = 0.06875$$

The absolute relative approximate error $|\in_a|$ at the ends of Iteration 3 is

$$|\in_a| = \left| \frac{x_m^{\text{new}} - x_m^{\text{old}}}{x_m^{\text{new}}} \right| \times 100$$
$$= \left| \frac{0.06875 - 0.0825}{0.06875} \right| \times 100$$
$$= 20\%$$

Still none of the significant digits are at least correct in the estimated root of the equation as the absolute relative approximate error is greater than 5%.
Seven more iterations were conducted and these iterations are shown in Table 1.

Table 1 Root of $f(x) = 0$ as function of number of iterations for bisection method.

Iteration	x_ℓ	x_u	x_m	$\lvert\in_a\rvert$%	$f(x_m)$
1	0.00000	0.11	0.055	----------	6.655×10^{-5}
2	0.055	0.11	0.0825	33.33	-1.622×10^{-4}
3	0.055	0.0825	0.06875	20.00	-5.563×10^{-5}
4	0.055	0.06875	0.06188	11.11	4.484×10^{-6}
5	0.06188	0.06875	0.06531	5.263	-2.593×10^{-5}
6	0.06188	0.06531	0.06359	2.702	-1.0804×10^{-5}
7	0.06188	0.06359	0.06273	1.370	-3.176×10^{-6}
8	0.06188	0.06273	0.0623	0.6897	6.497×10^{-7}
9	0.0623	0.06273	0.06252	0.3436	-1.265×10^{-6}
10	0.0623	0.06252	0.06241	0.1721	-3.0768×10^{-7}

At the end of 10^{th} iteration,

$$|\in_a| = 0.1721\%$$

Hence the number of significant digits at least correct is given by the largest value of m for which

$$|\in_a| \leq 0.5 \times 10^{2-m}$$

$$0.1721 \leq 0.5 \times 10^{2-m}$$

$$0.3442 \leq 10^{2-m}$$

$$\log(0.3442) \leq 2 - m$$

$$m \leq 2 - \log(0.3442) = 2.463$$

So

$$m = 2$$

The number of significant digits at least correct in the estimated root of 0.06241 at the end of the 10^{th} iteration is 2.