Background

What is the bisection method and what is it based on?

One of the first numerical methods developed to find the root of a nonlinear equation $f(x)=0$ was the bisection method (also called *binary-search* method). The method is based on the following theorem.

Theorem

An equation $f(x)=0$, where $f(x)$ is a real continuous function, has at least one root between x_ℓ and x_u if $f(x_\ell)f(x_u)<0$ (See Figure 1).

Note that if $f(x_\ell)f(x_u)>0$, there may or may not be any root between x_ℓ and x_u (Figures 2 and 3). If $f(x_\ell)f(x_u)<0$, then there may be more than one root between x_ℓ and x_u (Figure 4). So the theorem only guarantees one root between x_ℓ and x_u.

Bisection method

Since the method is based on finding the root between two points, the method falls under the category of bracketing methods.

Since the root is bracketed between two points, x_ℓ and x_u, one can find the mid-point, x_m between x_ℓ and x_u. This gives us two new intervals

1. x_ℓ and x_m, and
2. x_m and x_u.

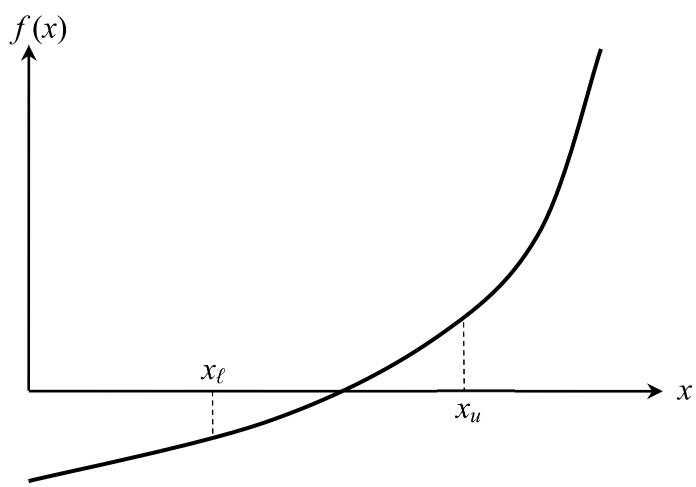

Figure 1 At least one root exists between the two points if the function is real, continuous, and changes sign.

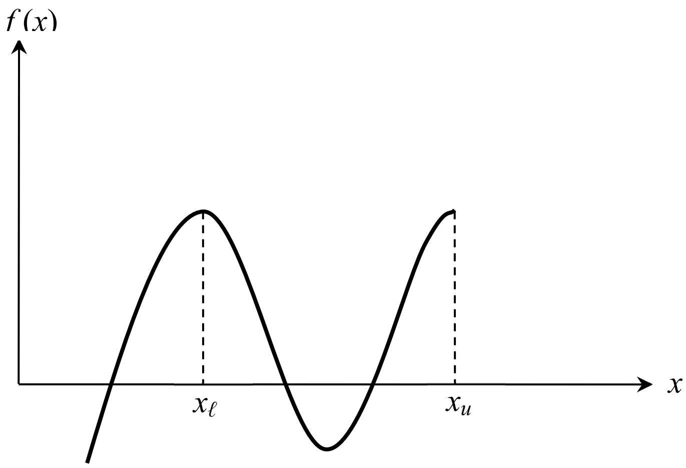

Figure 2 If the function $f(x)$ does not change sign between the two points, roots of the equation $f(x)=0$ may still exist between the two points.

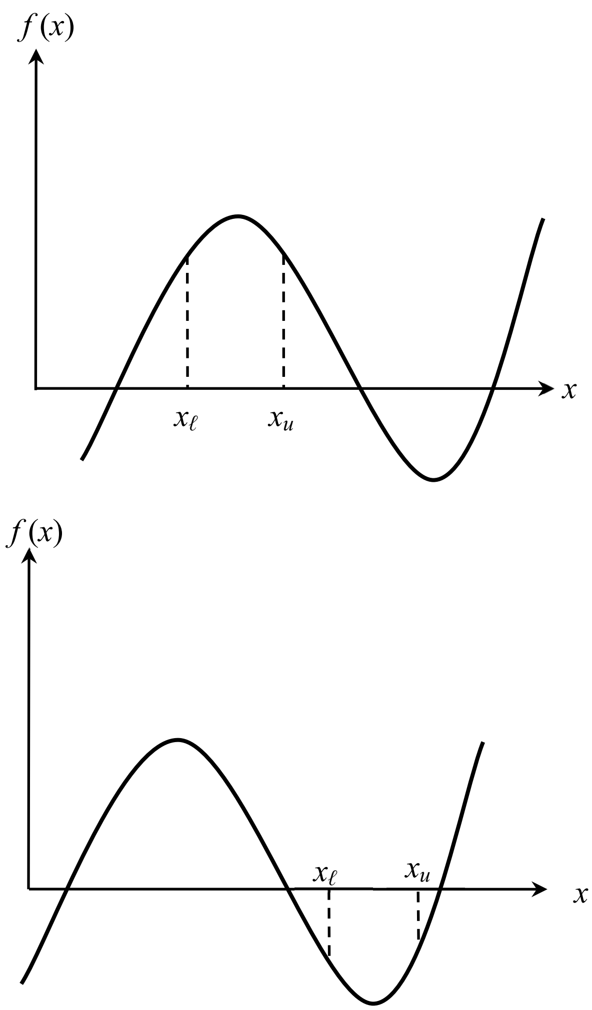

Figure 3 If the function $f(x)$ does not change sign between two points, there may not be any roots for the equation $f(x)=0$ between the two points.

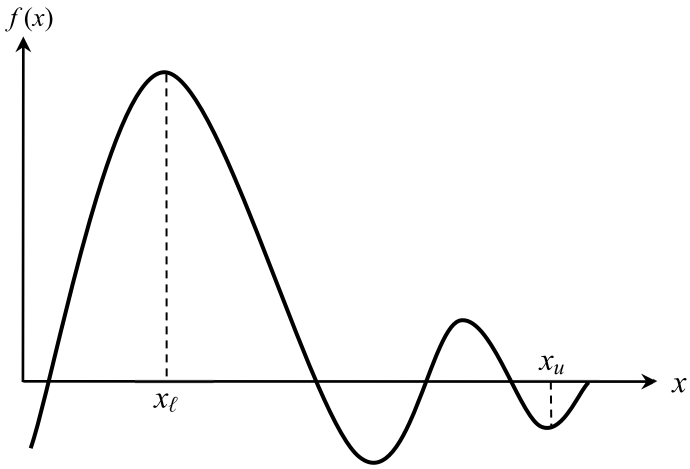

Figure 4 If the function $f(x)$ changes sign between the two points, more than one root for the equation $f(x)=0$ may exist between the two points.

Is the root now between x_ℓ and x_m or between x_m and x_u? Well, one can find the sign of $f(x_\ell)f(x_m)$, and if $f(x_\ell)f(x_m)<0$ then the new bracket is between x_ℓ and x_m, otherwise, it is between x_m and x_u. So, you can see that you are literally halving the interval. As one repeats this process, the width of the interval $[x_\ell, x_u]$ becomes smaller and smaller, and you can zero in to the root of the equation $f(x)=0$. The algorithm for the bisection method is given as follows.