Example 4

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30$$

Use the forward difference approximation of the second derivative of $v(t)$ to calculate the jerk at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2\,\text{s}$.

Solution

$$j(t_i) \approx \frac{v(t_{i+2}) - 2v(t_{i+1}) + v(t_i)}{(\Delta t)^2}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$
$$= 16 + 2$$
$$= 18$$

$$t_{i+2} = t_i + 2(\Delta t)$$
$$= 16 + 2(2)$$
$$= 20$$

$$j(16) \approx \frac{v(20) - 2v(18) + v(16)}{(2)^2}$$

$$v(20) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(20)}\right] - 9.8(20)$$
$$= 517.35\,\text{m/s}$$

$$v(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$
$$= 453.02\,\text{m/s}$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$
$$= 392.07\,\text{m/s}$$

$$j(16) \approx \frac{517.35 - 2(453.02) + 392.07}{4}$$
$$= 0.84515\,\text{m/s}^3$$

The exact value of $j(16)$ can be calculated by differentiating

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

twice as

$$a(t) = \frac{d}{dt}[v(t)] \text{ and}$$

$$j(t) = \frac{d}{dt}[a(t)]$$

Knowing that

$$\frac{d}{dt}[\ln(t)] = \frac{1}{t} \text{ and}$$

$$\frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{(14\times10^4 - 2100t)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

Similarly it can be shown that

$$j(t) = \frac{d}{dt}[a(t)]$$

$$= \frac{18000}{(-200+3t)^2}$$

$$j(16) = \frac{18000}{[-200+3(16)]^2}$$

$$= 0.77909\,\text{m/s}^3$$

The absolute relative true error is

$$|\in_t| = \left|\frac{0.77909 - 0.84515}{0.77909}\right| \times 100$$

$$= 8.4797\%$$

The formula given by Equation (5) is a forward difference approximation of the second derivative and has an error of the order of $O(\Delta x)$. Can we get a formula that has a better accuracy? Yes, we can derive the central difference approximation of the second derivative.

The Taylor series is

$$f(x_{i+1}) = f(x_i) + f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 + \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 + \ldots \quad (6)$$

where

$$x_{i+1} = x_i + \Delta x$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)\Delta x + \frac{f''(x_i)}{2!}(\Delta x)^2 - \frac{f'''(x_i)}{3!}(\Delta x)^3 + \frac{f''''(x_i)}{4!}(\Delta x)^4 - \ldots \quad (7)$$

where

$$x_{i-1} = x_i - \Delta x$$

Adding Equations (6) and (7), gives

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)(\Delta x)^2 + f''''(x_i)\frac{(\Delta x)^4}{12} + \ldots$$

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} - \frac{f''''(x_i)(\Delta x)^2}{12} + \ldots$$

$$= \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2} + O(\Delta x)^2$$