Example 3

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times 10^4}{14\times 10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30.$$

(a) Use the central difference approximation of the first derivative of $v(t)$ to calculate the acceleration at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2\,\text{s}$.

(b) Find the absolute relative true error for part (a).

Solution

$$a(t_i) \approx \frac{v(t_{i+1}) - v(t_{i-1})}{2\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$\begin{aligned} a(16) &\approx \frac{v(18) - v(14)}{2(2)} \\ &= \frac{v(18) - v(14)}{4} \end{aligned}$$

$$\begin{aligned} v(18) &= 2000\ln\left[\frac{14\times 10^4}{14\times 10^4 - 2100(18)}\right] - 9.8(18) \\ &= 453.02\,\text{m/s} \end{aligned}$$

$$\begin{aligned} v(14) &= 2000\ln\left[\frac{14\times 10^4}{14\times 10^4 - 2100(14)}\right] - 9.8(14) \\ &= 334.24\,\text{m/s} \end{aligned}$$

$$\begin{aligned} a(16) &\approx \frac{v(18) - v(14)}{4} \\ &= \frac{453.02 - 334.24}{4} \\ &= 29.694\,\text{m/s}^2 \end{aligned}$$

(b) The exact value of the acceleration at $t = 16\,\text{s}$ from Example 1 is

$$a(16) = 29.674\,\text{m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$\begin{aligned} |\epsilon_t| &= \left|\frac{29.674 - 29.694}{29.674}\right| \times 100 \\ &= 0.069157\% \end{aligned}$$

The results from the three difference approximations are given in Table 1.

Table 1 Summary of $a(16)$ using different difference approximations

Type of difference approximation	$a(16)$ (m/s^2)	$\lvert\in_t\rvert\%$
Forward	30.475	2.6967
Backward	28.915	2.5584
Central	29.695	0.069157

Clearly, the central difference scheme is giving more accurate results because the order of accuracy is proportional to the square of the step size. In real life, one would not know the exact value of the derivative – so how would one know how accurately they have found the value of the derivative? A simple way would be to start with a step size and keep on halving the step size until the absolute relative approximate error is within a pre-specified tolerance.

Take the example of finding $v'(t)$ for

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4-2100t}\right]-9.8t$$

at $t=16$ using the backward difference scheme. Given in Table 2 are the values obtained using the backward difference approximation method and the corresponding absolute relative approximate errors.

Table 2 First derivative approximations and relative errors for different Δt values of backward difference scheme.

Δt	$v'(t)$	$\lvert\in_a\rvert\%$
2	28.915	
1	29.289	1.2792
0.5	29.480	0.64787
0.25	29.577	0.32604
0.125	29.625	0.16355

From the above table, one can see that the absolute relative approximate error decreases as the step size is reduced. At $\Delta t = 0.125$, the absolute relative approximate error is 0.16355%, meaning that at least 2 significant digits are correct in the answer.