Example 1

The velocity of a rocket is given by

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t,\ 0 \le t \le 30$$

where v is given in m/s and t is given in seconds. At $t = 16\,\text{s}$,

a) use the forward difference approximation of the first derivative of $v(t)$ to calculate the acceleration. Use a step size of $\Delta t = 2\,\text{s}$.

b) find the exact value of the acceleration of the rocket.

c) calculate the absolute relative true error for part (b).

Solution

(a) $$a(t_i) \approx \frac{v(t_{i+1}) - v(t_i)}{\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$t_{i+1} = t_i + \Delta t$$
$$= 16 + 2$$
$$= 18$$

$$a(16) \approx \frac{v(18) - v(16)}{2}$$

$$v(18) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18)$$
$$= 453.02\ \text{m/s}$$

$$v(16) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(16)}\right] - 9.8(16)$$
$$= 392.07\ \text{m/s}$$

Hence

$$a(16) \approx \frac{v(18) - v(16)}{2}$$
$$= \frac{453.02 - 392.07}{2}$$
$$= 30.474\ \text{m/s}^2$$

(b) The exact value of $a(16)$ can be calculated by differentiating

$$v(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t$$

as

$$a(t) = \frac{d}{dt}\left[v(t)\right]$$

Knowing that

$$\frac{d}{dt}\left[\ln(t)\right] = \frac{1}{t} \text{ and } \frac{d}{dt}\left[\frac{1}{t}\right] = -\frac{1}{t^2}$$

$$a(t) = 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)\frac{d}{dt}\left(\frac{14\times10^4}{14\times10^4 - 2100t}\right) - 9.8$$

$$= 2000\left(\frac{14\times10^4 - 2100t}{14\times10^4}\right)(-1)\left(\frac{14\times10^4}{\left(14\times10^4 - 2100t\right)^2}\right)(-2100) - 9.8$$

$$= \frac{-4040 - 29.4t}{-200 + 3t}$$

$$a(16) = \frac{-4040 - 29.4(16)}{-200 + 3(16)}$$

$$= 29.674\,\text{m/s}^2$$

(c) The absolute relative true error is

$$\left|\in_t\right| = \left|\frac{\text{True Value} - \text{Approximate Value}}{\text{True Value}}\right| \times 100$$

$$= \left|\frac{29.674 - 30.474}{29.674}\right| \times 100$$

$$= 2.6967\%$$