Example 3

The velocity of a rocket is given by

$$\nu(t) = 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100t}\right] - 9.8t, 0 \le t \le 30.$$

(a) Use the central difference approximation of the first derivative of $\nu(t)$ to calculate the acceleration at $t = 16\,\text{s}$. Use a step size of $\Delta t = 2\,\text{s}$.

(b) Find the absolute relative true error for part (a).

Solution

$$a(t_i) \approx \frac{\nu(t_{i+1}) - \nu(t_{i-1})}{2\Delta t}$$

$$t_i = 16$$

$$\Delta t = 2$$

$$\begin{aligned} t_{i+1} &= t_i + \Delta t \\ &= 16 + 2 \\ &= 18 \end{aligned}$$

$$\begin{aligned} t_{i-1} &= t_i - \Delta t \\ &= 16 - 2 \\ &= 14 \end{aligned}$$

$$\begin{aligned} a(16) &\approx \frac{\nu(18) - \nu(14)}{2(2)} \\ &= \frac{\nu(18) - \nu(14)}{4} \end{aligned}$$

$$\begin{aligned} \nu(18) &= 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(18)}\right] - 9.8(18) \\ &= 453.02\,\text{m/s} \end{aligned}$$

$$\begin{aligned} \nu(14) &= 2000\ln\left[\frac{14\times10^4}{14\times10^4 - 2100(14)}\right] - 9.8(14) \\ &= 334.24\,\text{m/s} \end{aligned}$$

$$\begin{aligned} a(16) &\approx \frac{\nu(18) - \nu(14)}{4} \\ &= \frac{453.02 - 334.24}{4} \\ &= 29.694\,\text{m/s}^2 \end{aligned}$$

(b) The exact value of the acceleration at $t = 16\,\text{s}$ from Example 1 is

$$a(16) = 29.674\ \text{m/s}^2$$

The absolute relative true error for the answer in part (a) is

$$|\in_t| = \left|\frac{29.674 - 29.694}{29.674}\right| \times 100$$

$$= 0.069157\%$$

The results from the three difference approximations are given in Table 1.

Table 1 Summary of $a(16)$ using different difference approximations

Type of difference approximation	$a(16)$ (m/s^2)	$\|\in_t\|\%$
Forward	30.475	2.6967
Backward	28.915	2.5584
Central	29.695	0.069157

Clearly, the central difference scheme is giving results that are more accurate because the order of accuracy is proportional to the square of the step size.